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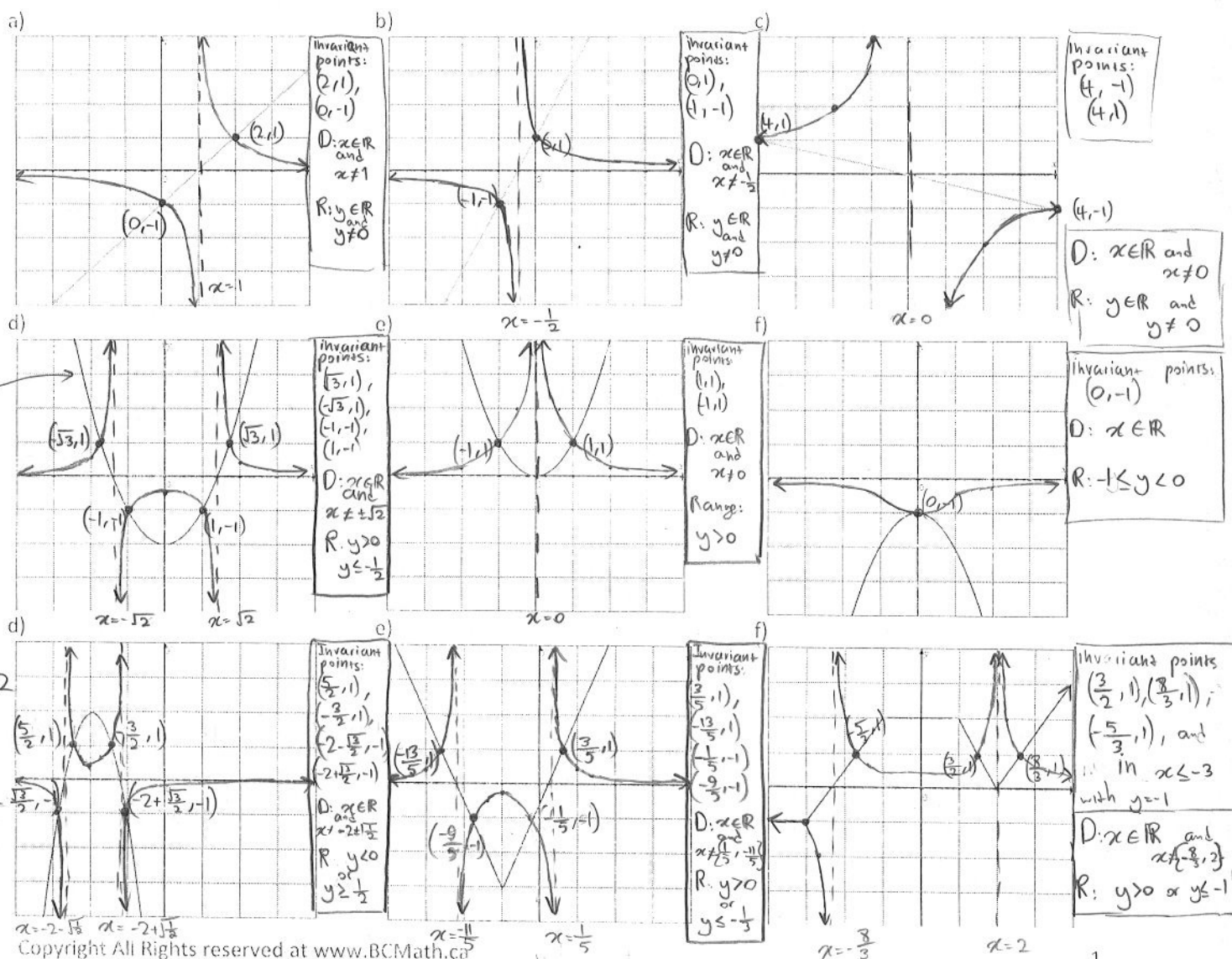
Date: Nov 25th 2022

Math 10/11 Enriched: Section 2.7 Reciprocal of Quadratic Functions

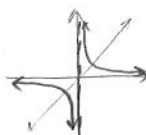
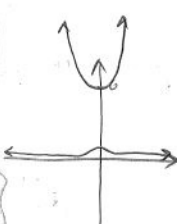
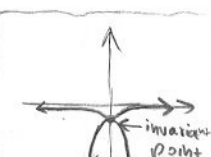
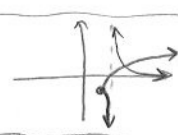
1. Given the equation for $y = f(x)$, find the equation for $y = \frac{1}{f(x)}$

a) $y = 3x - 5$ $y = \frac{1}{f(x)} = \frac{1}{3x-5}$	b) $y = \frac{2x-1}{3}$ $y = \frac{1}{f(x)} = \frac{3}{2x-1}$	c) $y = \frac{3x-5}{5x-1}$ $y = \frac{1}{f(x)} = \frac{5x-1}{3x-5}$	d) $y = 3x^2 + 4$ $y = \frac{1}{f(x)} = \frac{1}{3x^2+4}$
e) $y = 3$ $y = \frac{1}{f(x)} = \frac{1}{3}$	f) $x = -5$	g) $y = -5x^2 - 6$ $y = \frac{1}{f(x)} = \frac{1}{-5x^2-6}$	h) $y = 5x^3 - 7x^2 + 22 - 6x$ $y = \frac{1}{f(x)} = \frac{1}{5x^3-7x^2-6x+22}$

2. Graph the reciprocal of each function: Indicate the equations of the asymptotes, coordinate of the invariant points, domain and range of the reciprocal function.

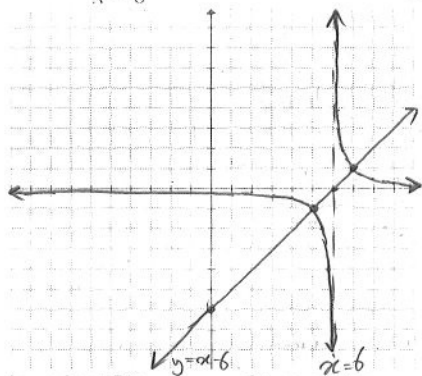


3. Indicate the coordinates of the invariant points, the equations of the vertical asymptotes, and the domain and range for each of the following functions below:

a) $y = \frac{1}{x}$ Equation of vertical asymptote: $x=0$ Invariant points: $(1,1)$ $(-1,-1)$ Domain: $x \in \mathbb{R} \cap x \neq 0$ Range: $y \in \mathbb{R} \cap y \neq 0$ 	b) $y = \frac{1}{3x-5}$ Asymptote: $3x-5 \neq 0$ $3x \neq 5$ $x \neq \frac{5}{3}$ Domain: $x \in \mathbb{R} \cap x \neq \frac{5}{3}$ Range: $y \in \mathbb{R} \cap y \neq 0$ Invariant points: $1 = \frac{1}{3x-5}$ $3x-5=1$ $3x=6$ $x=2$ $-1 = \frac{1}{3x-5}$ $3x-5=-1$ $3x=4$ $x=\frac{4}{3}$ $(x,y) = (2,1), (\frac{4}{3}, -1)$	c) $y = \frac{1}{-\frac{5}{2}x-11}$ Asymptote: $-\frac{5}{2}x-11 \neq 0$ $-\frac{5}{2}x \neq 11$ $x \neq -\frac{22}{5}$ Domain: $x \in \mathbb{R} \cap x \neq -\frac{22}{5}$ Range: $y \in \mathbb{R} \cap y \neq 0$ Invariant points: $\frac{1}{-\frac{5}{2}x-11} = 1 \Rightarrow -\frac{5}{2}x-11=1$ $-\frac{5}{2}x=12$ $x=-\frac{24}{5}$ $\frac{1}{-\frac{5}{2}x-11} = -1 \Rightarrow \frac{5}{2}x+11=1$ $\frac{5}{2}x=-10$ $x=-4$ $(x,y) = (-\frac{24}{5}, 1), (-4, -1)$
d) $y = \frac{1}{5x^2+6}$ No asymptote, parabola has no roots! No invariant points! Domain: $x \in \mathbb{R}$ Range: vertex: $(0, \frac{1}{6})$ $0 < y \leq \frac{1}{6}$ 	e) $y = \frac{1}{-(x-3)^2+4}$ Asymptotes: $-(x-3)^2+4 \neq 0$ $(x-3)^2 \neq 4$ $x \neq 5, 1$ Domain: $x \in \mathbb{R} \cap x \neq 1, 5$ Range: vertex: $(3, 4)$ $y < 0$ $y \geq \frac{1}{4}$ Equations: $x=5$ $x=1$ Invariant points: $-(x-3)^2+4=1$ $(x-3)^2=3$ $x=3 \pm \sqrt{3}$ $-(x-3)^2+4=-1$ $(x-3)^2=5$ $x=3 \pm \sqrt{5}$ $(x,y) = (3+\sqrt{3}, 1), (3-\sqrt{3}, 1), (3+\sqrt{5}, -1), (3-\sqrt{5}, -1)$	f) $y = \frac{1}{-3x^2-1}$ Asymptotes: $-3x^2-1 \neq 0$ $x^2 \neq -\frac{1}{3}$ Domain: $x \in \mathbb{R}$ Range: $-1 \leq y < 0$ No asymptotes Invariant points: $-3x^2-1=-1$ $-3x^2=0$ $x^2=0$ $x=0$ $(x,y) = (0, -1)$ 
g) $y = \frac{1}{-2(x-7)^2+16}$ Asymptotes: $-2(x-7)^2+16 \neq 0$ $(x-7)^2 \neq 8$ $x \neq 7 \pm 2\sqrt{2}$ Domain: $x \in \mathbb{R} \cap x \neq 7 \pm 2\sqrt{2}$ Range: $y < 0$ or $y \geq \frac{1}{16}$ Equations of vertical asymptotes: $x=7+2\sqrt{2}, x=7-2\sqrt{2}$ Invariant points: $-2(x-7)^2+16=1$ $(x-7)^2 = \frac{15}{2}$ $x=7 \pm \sqrt{\frac{15}{2}}$ $-2(x-7)^2+16=-1$ $(x-7)^2 = \frac{17}{2}$ $x=7 \pm \sqrt{\frac{17}{2}}$ $(x,y) = (7+\sqrt{\frac{15}{2}}, 1), (7-\sqrt{\frac{15}{2}}, 1), (7+\sqrt{\frac{17}{2}}, -1), (7-\sqrt{\frac{17}{2}}, -1)$	h) $y = \frac{1}{-1+\sqrt{3x-5}}$ Asymptotes: $-1+\sqrt{3x-5} \neq 0$ $\sqrt{3x-5} \neq 1$ $3x-5 \neq 4$ $x \neq 3$ Domain: $x \geq \frac{5}{3}$ or $x \neq 3$ Range: $y > 0$ or $y \leq -1$ Equation: $x=2$ Invariant points: $-1+\sqrt{3x-5}=1$ $\sqrt{3x-5}=2$ $3x-5=4$ $3x=9$ $x=3$ $-1+\sqrt{3x-5}=-1$ $\sqrt{3x-5}=0$ $3x-5=0$ $3x=5$ $x=\frac{5}{3}$ $(x,y) = (3, 1), (\frac{5}{3}, -1)$ 	i) $y = \frac{2}{(x-3)(x+4)}$ Asymptotes: $(x-3)(x+4) \neq 0$ $x \neq 3$ $x \neq -4$ Domain: $x \in \mathbb{R} \cap x \neq 3, -4$ Range: $y > 0$ or $y < -\frac{8}{49}$ Equations: $x=3, x=-4$ Invariant points: $1 = \frac{2}{(x-3)(x+4)}$ $(x-3)(x+4)=2$ $x^2+x-12=2$ $x^2+x-14=0$ $x = \frac{-1 \pm \sqrt{1+56}}{2} = \frac{-1 \pm \sqrt{57}}{2}$ $-1 = \frac{2}{(x-3)(x+4)}$ $(x-3)(x+4)=-2$ $x^2+x-12=-2$ $x^2+x-10=0$ $x = \frac{-1 \pm \sqrt{1+40}}{2} = \frac{-1 \pm \sqrt{41}}{2}$ $(x,y) = (\frac{-1+\sqrt{57}}{2}, 1), (\frac{-1-\sqrt{57}}{2}, 1), (\frac{-1+\sqrt{41}}{2}, -1), (\frac{-1-\sqrt{41}}{2}, -1)$

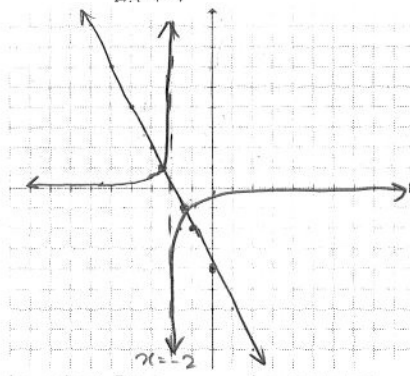
4. Graph each of the following reciprocal functions. Indicate the asymptotes, coordinates of the invariant points, domain and range:

a) $y = \frac{1}{x-6}$



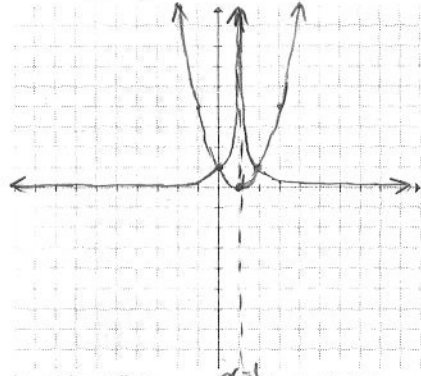
Invariant Pts: $(7,1), (5,-1)$
 Asymptotes: $x=6$
 Domain: $x \in \mathbb{R} \cap x \neq 6$
 Range: $y \in \mathbb{R} \cap y \neq 0$

b) $y = \frac{1}{2x+4}$



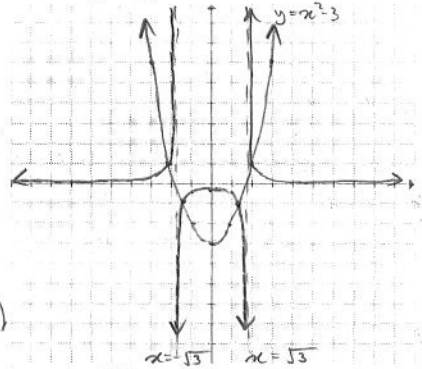
Invariant Pts: $(\frac{5}{2}, 1), (-\frac{3}{2}, -1)$
 Asymptotes: $x=-2$
 Domain: $x \in \mathbb{R}, x \neq -2$
 Range: $y \in \mathbb{R}, y \neq 0$

b) $y = \frac{1}{(x-1)^2}$



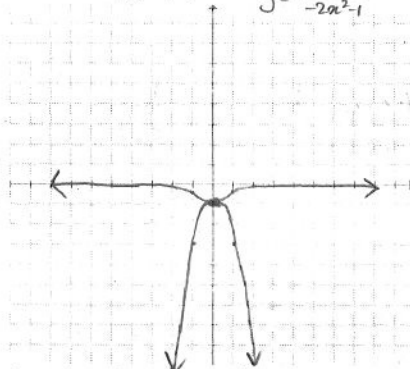
Invariant Pts: $(0,1), (2,1)$
 Asymptotes: $x=1$
 Domain: $x \in \mathbb{R}, x \neq 1$
 Range: $y > 0$

a) $y = \frac{1}{x^2-3}$



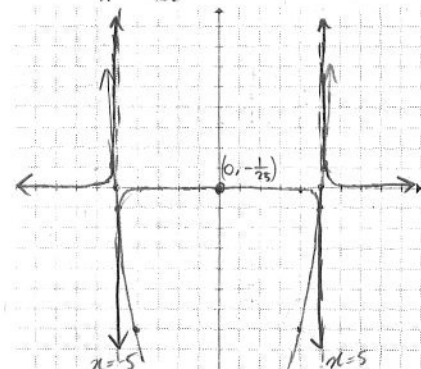
Invariant Pts: $(2,1), (-2,1), (\sqrt{2}, -1), (-\sqrt{2}, -1)$
 Asymptotes: $x=\sqrt{3}, x=-\sqrt{3}$
 Domain: $x \in \mathbb{R} \cap x \neq \pm\sqrt{3}$
 Range: $y > 0 \text{ or } y \leq -\frac{1}{3}$

b) $y = \frac{1}{2x^2+1}$



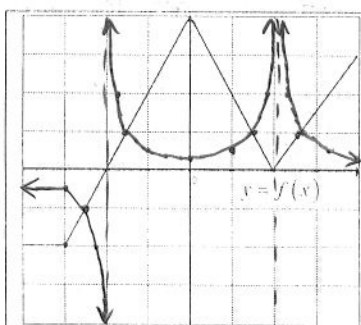
Invariant Pts: $(0,-1)$
 Asymptotes: None
 Domain: $x \in \mathbb{R}$
 Range: $-1 \leq y < 0$

b) $y = \frac{1}{x^2-25}$



Invariant Pts: $(26,1), (-26,1), (5,1), (-5,-1)$
 Asymptotes: $x=5, x=-5$
 Domain: $x \in \mathbb{R} \cap x \neq \pm 5$
 Range: $y > 0 \text{ or } y \leq -\frac{1}{25}$

5. Given the graph of $y = f(x)$, and $g(x) = \frac{1}{f(x)}$ find the following values:



a) $g(0)$
 $g(0) = \frac{1}{f(0)} = \frac{1}{4}$

b) $g(2)$
 $g(2) = \frac{1}{f(2)} = \frac{1}{0} = \text{undefined}$

c) $g(1)$
 $g(1) = \frac{1}{f(1)} = \frac{1}{2}$

d) $g(k) = 1$
 $g(k) = \frac{1}{f(k)} = 1 \Rightarrow f(k) = 1$
 $k = \{2, 3, 5\}$

e) $g(-3)$
 $g(-3) = \frac{1}{f(-3)} = -\frac{1}{2}$

f) $g(1)f(1)$
 $g(1) = \frac{1}{f(1)} = \frac{1}{2}$
 $f(1) = 2$
 $f(1)g(1) = 2 \times \frac{1}{2} = 1$

g) $g(4) \times f(4)$
 $g(4) = \frac{1}{f(4)}$
 $\frac{1}{f(4)} \times f(4) = 1$

h) $g(4) \div f(4)$
 $g(4) = \frac{1}{f(4)}$
 $\frac{g(4)}{f(4)} = \frac{\frac{1}{f(4)}}{f(4)} = \frac{1}{f(4)^2} = \frac{1}{9}$

i) $g(-2) \times f(-2)$
 $g(-2) = \frac{1}{f(-2)} = \frac{1}{0} = \text{undefined}$
 $\emptyset$

6. Given the function $y = \frac{1}{(x-3)^2 + k}$, for what values of "k" will the graph not have any invariant points?

$$\begin{aligned} (x-3)^2 + k &= 1 \\ (x-3)^2 &= 1-k \end{aligned}$$

In order to have no invariant points, the equation above must have no solutions.

$$1-k < 0$$

$$k > 1$$

$$\begin{aligned} (x-3)^2 + k &= -1 \\ (x-3)^2 &= -1-k \\ -1-k &< 0 \\ k &> -1 \end{aligned}$$

NOTE: this operation was not needed

$k > 1$



7. Given the function $y = \frac{-1}{(x-2)^2 + k}$, for what values of "k" will the graph have four invariant points?

$$y = \frac{-1}{(x-2)^2 + k} = \frac{1}{-(x-2)^2 - k}$$

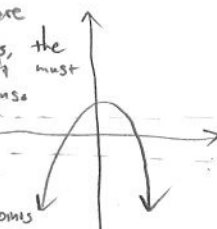
$$-(x-2)^2 - k = 1$$

$$(x-2)^2 = -k-1$$

$$-k-1 > 0$$

$k < -1$

In order for there to be 4 invariant points, the equation on the left must have two solutions. Since the graph opens downward, if there are 2 invariant points at $y=1$, there must be 2 invariant points at $y=-1$.



8. Given the function $y = \frac{-1}{(x+2)^2 + 4 - k}$, for what values of "k" will the graph not have any vertical asymptotes?

$$(x+2)^2 + 4 - k = 0$$

the equation must have no roots if there aren't going to be any vertical asymptotes.

$\Rightarrow$

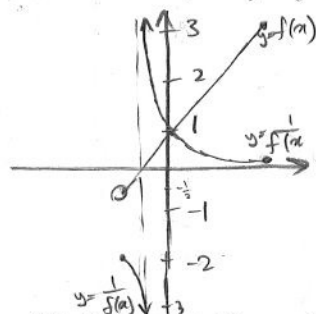
$$(x+2)^2 = k-4$$

$$k-4 < 0$$

$k < 4$

For the equation to have no solutions, $x+2 = \pm\sqrt{k-4}$, $k-4$ must be negative

9. If the range of $y = f(x)$ is $-0.5 < y \leq 3$, what is the range of $y = \frac{1}{f(x)}$? Justify your answer



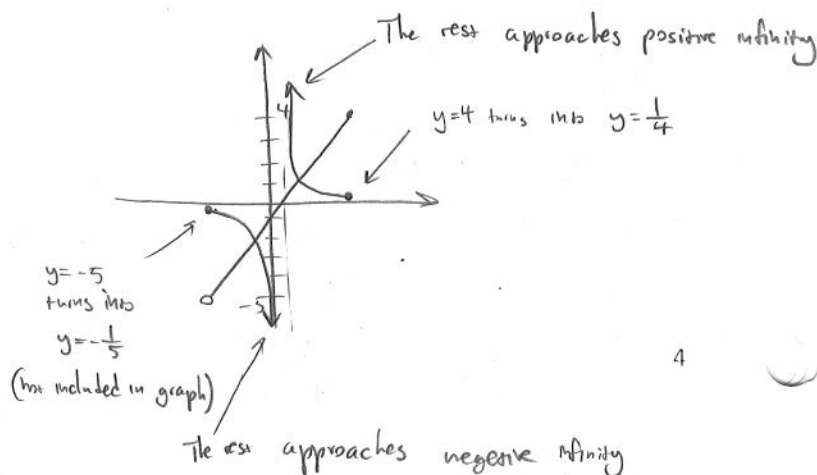
Range of $y = \frac{1}{f(x)}$: $y < -2$ or $y \geq \frac{1}{3}$

$y \geq \frac{1}{3}$ since in $f(x)$, the maximum "y" value is 3. See graph for more justification.

$y < -2$ because the minimum "y"-value is $-\frac{1}{2}$. $-\frac{1}{2} \Rightarrow -2$. See graph for more justification.

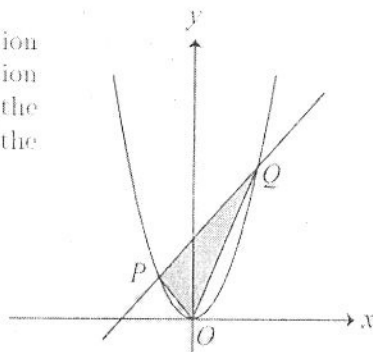
10. If the range of $y = f(x)$ is $-5 < y \leq 4$, what is the range of $y = \frac{1}{f(x)}$? Justify your answer

Range of $y = \frac{1}{f(x)}$: $y < -\frac{1}{5}$ or $y \geq \frac{1}{4}$



Suppose that $k > 0$ and that the line with equation $y = 3kx + 4k^2$ intersects the parabola with equation $y = x^2$ at points P and Q , as shown. If O is the origin and the area of $\triangle OPQ$ is 80, then the slope of the line is

- (A) 4 (B) 3 (C) $\frac{15}{4}$
(D) 6 (E) $\frac{21}{4}$



$$y = 3kx + 4k^2$$

$$y = x^2$$

$$x^2 = 3kx + 4k^2$$

$$x^2 - 3kx - 4k^2 = 0$$

$$x = \frac{3k \pm \sqrt{9k^2 + 16k^2}}{2} = \frac{3k \pm 5k}{2}$$

$$x_1 = 4k \quad y_1 = 16k^2$$

$$x_2 = -k \quad y_2 = k^2$$

$$\triangle OPQ : (0,0), (-k, k^2), (4k, 16k^2)$$

Determinant:

$$4k^3 - k \times 16k^2 - 16k^3$$

$$\text{Area: } \frac{4k^3 + 16k^3}{2} = 10k^3$$

$$10k^3 = 80$$

$$k^3 = 8$$

$$k = 2$$

Consider the quadratic equation $x^2 - (r+7)x + r+87 = 0$ where r is a real number. This equation has two distinct real solutions x which are both negative exactly when $p < r < q$, for some real numbers p and q . The value of $p^2 + q^2$ is

- (A) 7618 (B) 698 (C) 1738 (D) 7508 (E) 8098

$$x^2 - (r+7)x + r+87 = 0$$

Roots:

$$x = \frac{r+7 \pm \sqrt{(r+7)^2 - 4(r+87)}}{2} = \frac{r+7 \pm \sqrt{r^2 + 14r + 49 - 4r - 348}}{2} = \frac{r+7 \pm \sqrt{r^2 + 10r - 299}}{2}$$

If we want both roots to be negative, we must have $-(r+7)$ be less than $-\sqrt{r^2 + 10r - 299}$.

This can be solved algebraically, but you have to be mindful that $r > -87$.

$$r+7 < -\sqrt{r^2 + 10r - 299}$$

$$r > -87$$

Since we know that $r > -87$ and $r < -23$, we can conclude that $p = -87, q = -23$.

$$p^2 + q^2 = 8098$$

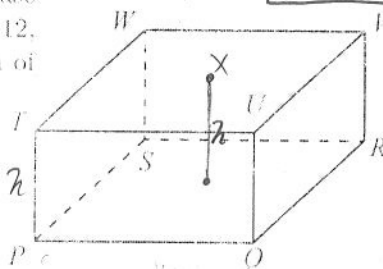
$$+13 < r \text{ or } r < -23$$

$r+7 < 0$ since if $(r+7)$ is positive, $\frac{r+7 \pm \sqrt{r^2 + 10r - 299}}{2}$ will also be positive.

$$r < -23$$

Rectangular prism $PQRSTU$ has a square base $PQRS$. Point X is on the face $TUVW$ so that $PX = 12$, $QX = 10$ and $RX = 8$. The maximum possible area of rectangle $PQUT$ is closest to

- (A) 67.81 (B) 67.82 (C) 67.90
(D) 67.86 (E) 67.88



$$S-a = \sqrt{a^2 + 36}$$

$$S^2 - 2aS + a^2 = a^2 + 36$$

$$a = \frac{S^2 + 36}{2S}$$

$$S = \sqrt{144 - h^2 - a^2} + \sqrt{100 - h^2 - a^2}$$

$$S - \sqrt{144 - h^2 - a^2} = \sqrt{100 - h^2 - a^2}$$

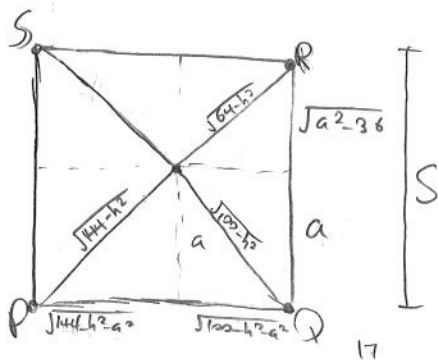
square both sides & cancel

$$S^2 + 44 = 2S\sqrt{144 - h^2 - a^2}$$

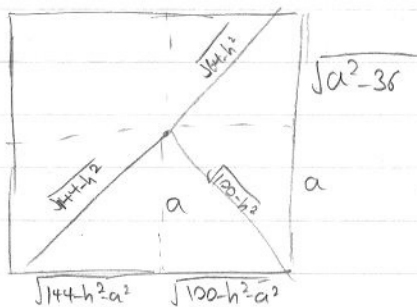
$$S^2 h^2 = -\frac{1}{2} S^4 + 104 S^2 - 808$$

$$\text{MAX: } \sqrt{4600} = 67.82$$

See separate paper for full solution



Box Q. Solution (Last Q. on 2.7)



$$S - a = \sqrt{a^2 - 36}$$

$$S^2 - 2aS + a^2 = a^2 - 36$$

$$S^2 + 36 = 2aS$$

$$a = \frac{S^2 + 36}{2S}$$

$$S = \sqrt{144 - h^2 - a^2} + \sqrt{100 - h^2 - a^2}$$

$$S - \sqrt{144 - h^2 - a^2} = \sqrt{100 - h^2 - a^2}$$

$$S^2 + 144 - h^2 - a^2 - 2S\sqrt{144 - h^2 - a^2} = 100 - h^2 - a^2$$

$$S^2 + 44 = 2S\sqrt{144 - h^2 - a^2}$$

$$(S^2 + 44)^2 = 4S^2(144 - h^2 - a^2)$$

$$S^4 + 88S^2 + 44^2 = 4(144)S^2 - 4S^2h^2 - 4S^2a^2$$

$$S^4 + 88S^2 + 44^2 = 4(144)S^2 - 4S^2h^2 - 4S^2\left(\frac{(S^2 + 36)^2}{4S^2}\right)$$

$$\begin{array}{rcll} S^4 + 88S^2 + 44^2 & = & 4(144)S^2 - 4S^2h^2 - (S^2 + 36)^2 \\ 2S^4 & & 3232 & 488S^2 & -8^4 - 128S^2 - 36^2 \\ & & & 416S^2 & \end{array}$$

$$2S^4 + 4S^2h^2 - 416S^2 + 3232 = 0$$

$$S^4 + 2S^2h^2 - 208S^2 + 1616 = 0$$

$$2S^2h^2 = -S^4 + 208S^2 - 1616$$

$$S^2h^2 = -\frac{1}{2}S^4 + 104S^2 - 808$$

$$S^2h^2 = -\frac{1}{2}B^2 + 104B - 808$$

$$S^2h^2 = -\frac{1}{2}(B^2 - 208B + 1616)$$

$$S^2h^2 = -\frac{1}{2}[B^2 - 208B + (104)^2] + 4600$$

$$S^2 = B$$

$$\text{Max: } \sqrt{4600} = 67.82$$